

LE MATEMATICHE

Vol. LXVIII (2013) – Fasc. II, pp. 141–158

doi: 10.4418/2013.68.2.11

DIFFERENTIAL SANDWICH THEOREMS FOR p -VALENT FUNCTIONS ASSOCIATED WITH GENERALIZED MULTIPLIER TRANSFORMATIONS

RABHA M. EL-ASHWAH - MOHAMED K. AOUF
ALI SHAMANDY - SHEZA M. EL-DEEB

In this paper, we obtain some applications of theory of differential subordination, superordination and sandwich results involving the operator $\mathcal{J}_p^m(\lambda, \ell)$.

1. Introduction

Let $H(U)$ denote the class of analytic functions in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$ and $H[a, p]$ denote the subclass of functions $f \in H(U)$ of the form:

$$f(z) = a + a_p z^p + a_{p+1} z^{p+1} + \dots \quad (a \in \mathbb{C}; p \in \mathbb{N} = \{1, 2, \dots\}).$$

Also, let $\mathcal{A}(p)$ denote the subclass of functions $f \in H(U)$ of the form:

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \quad (p \in \mathbb{N}). \quad (1)$$

Also, let $\mathcal{A}(1) = \mathcal{A}$.

If f and g are analytic function in U , we say that f is subordinate to g , written $f \prec g$ if there exists a Schwarz function w , which is analytic in U with

Entrato in redazione: 26 ottobre 2012

AMS 2010 Subject Classification: 30C45.

Keywords: p -Valent functions, Differential subordination, Superordination, Sandwich theorems.

$w(0) = 0$ and $|w(z)| < 1$ for all $z \in U$, such that $f(z) = g(w(z))$. Furthermore, if the function g is univalent in U , then we have the following equivalence (see [11] and [19]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

For $k, h \in H(U)$, let $\varphi(r, s, t; z) : \mathbb{C}^3 \times U \rightarrow \mathbb{C}$ and let h be univalent in U . If $k(z)$ satisfies the second order differential subordination

$$\varphi(k(z), zk'(z), z^2k''(z); z) \prec h(z), \quad (2)$$

then $k(z)$ is a solution of the differential subordination (2). The univalent function $q(z)$ is called a dominant of the solutions of the differential subordination, if $k(z) \prec q(z)$ for all the functions $k(z)$ satisfying (2). A dominant $\tilde{q}(z)$ is said to be the best dominant of (2) if $\tilde{q}(z) \prec q(z)$ for all dominants $q(z)$. If $k(z)$ and $\varphi(k(z), zk'(z), z^2k''(z); z)$ are univalent functions in U and if $k(z)$ satisfies the second order differential superordination

$$h(z) \prec \varphi(k(z), zk'(z), z^2k''(z); z), \quad (3)$$

then $k(z)$ is a solution of the differential superordination (3). The univalent function $q(z)$ is called a subordinator of the solutions of the differential superordination, if $q(z) \prec k(z)$ for all the functions $k(z)$ satisfying (3). A subordinator $\tilde{q}(z)$ is said to be the best subordinator of (3) if $q(z) \prec \tilde{q}(z)$ for all the subordinants $q(z)$. Recently Miller and Mocanu [20] obtained conditions on the functions h, q and φ for which the following implication holds:

$$h(z) \prec \varphi(k(z), zk'(z), z^2k''(z); z) \Rightarrow q(z) \prec k(z).$$

Using the results of Miller and Mocanu [20], Bulboacă [10] considered certain classes of first order differential subordinations as well as superordination-preserving integral operators [9]. Ali et al. [1], have used the results of Bulboacă [10] (see also [3] and [4]) to obtain sufficient conditions for normalized analytic functions to satisfy:

$$q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),$$

where q_1 and q_2 are univalent functions in U with $q_1(0) = q_2(0) = 1$.

Prajapat [24] defined a generalized multiplier transformation operator, as follows:

$$\mathcal{J}_p^m(\lambda, \ell) : \mathcal{A}(p) \rightarrow \mathcal{A}(p)$$

$$\mathcal{J}_p^m(\lambda, \ell) f(z) = z^p + \sum_{k=p+1}^{\infty} \left(\frac{p+\ell+\lambda(k-p)}{p+\ell} \right)^m a_k z^k$$

$$(\lambda \geq 0; \ell > -p; p \in \mathbb{N}; m \in \mathbb{Z} = \{0, \pm 1, \dots\}; z \in U). \quad (4)$$

It is readily verified from (4) that

$$\begin{aligned} & \lambda z (\mathcal{J}_p^m(\lambda, \ell) f(z))' \\ &= (\ell + p) \mathcal{J}_p^{m+1}(\lambda, \ell) f(z) - [\ell + p(1 - \lambda)] \mathcal{J}_p^m(\lambda, \ell) f(z) \quad (\lambda > 0). \end{aligned} \quad (5)$$

By specializing the parameters m , λ , ℓ and p , we obtain the following operators studied by various authors:

- (i) $\mathcal{J}_p^m(\lambda, \ell) f(z) = I_p^m(\lambda, \ell) f(z) \quad (\ell \geq 0, p \in \mathbb{N}, \lambda \geq 0 \text{ and } m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ (see [[12]]);
- (ii) $\mathcal{J}_p^m(1, \ell) f(z) = I_p(m, \ell) f(z) \quad (\ell \geq 0, p \in \mathbb{N} \text{ and } m \in \mathbb{N}_0)$ (see [18] and [29]);
- (iii) $\mathcal{J}_p^m(\lambda, 0) f(z) = D_{\lambda, p}^m f(z) \quad (\lambda \geq 0, p \in \mathbb{N} \text{ and } m \in \mathbb{N}_0)$ (see [5]);
- (iv) $\mathcal{J}_p^m(1, 0) f(z) = D_p^m f(z) \quad (m \in \mathbb{N}_0 \text{ and } p \in \mathbb{N})$ (see [6], [17] and [21]);
- (v) $\mathcal{J}_p^{-m}(\lambda, \ell) f(z) = J_p^m(\lambda, \ell) f(z) \quad (\ell \geq 0, \lambda \geq 0, p \in \mathbb{N} \text{ and } m \in \mathbb{N}_0)$ (see [7], [15] and [28]);
- (vi) $\mathcal{J}_p^{-m}(1, 1) f(z) = D^m f(z) \quad (m \in \mathbb{Z})$ (see [23]);
- (vii) $\mathcal{J}_1^m(1, \ell) f(z) = I_\ell^m f(z) \quad (\ell \geq 0 \text{ and } m \in \mathbb{N}_0)$ (see [13] and [14]);
- (viii) $\mathcal{J}_1^m(\lambda, 0) f(z) = D_\lambda^m f(z) \quad (\lambda \geq 0 \text{ and } m \in \mathbb{N}_0)$ (see [2]);
- (ix) $\mathcal{J}_1^m(1, 0) f(z) = D^m f(z) \quad (m \in \mathbb{N}_0)$ (see [26]);
- (x) $\mathcal{J}_1^{-m}(\lambda, 0) f(z) = I_\lambda^{-m} f(z) \quad (\lambda \geq 0 \text{ and } m \in \mathbb{N}_0)$ (see [22] and [8]);
- (xi) $\mathcal{J}_1^{-m}(1, 1) f(z) = I^m f(z) \quad (m \in \mathbb{N}_0)$ (see [16]).

2. Definitions and preliminaries

In order to prove our results, we shall need the following definition and lemmas.

Definition 2.1 ([20]). Let \mathcal{Q} be the set of all functions f that are analytic and injective on $\overline{U} \setminus E(f)$, where $E(f) = \{\zeta \in \partial U : \lim_{z \rightarrow \zeta} f(z) = \infty\}$ and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial U \setminus E(f)$.

Lemma 2.2 ([19]). *Let q be univalent in the unit disc U and let θ and ϕ be analytic in a domain D containing $q(U)$, with $\phi(w) \neq 0$ when $w \in q(U)$. Set*

$$Q(z) = zq'(z)\phi(q(z)) \text{ and } h(z) = \theta(q(z)) + Q(z), \quad (6)$$

suppose that

(i) Q is a starlike function in U ,

(ii) $\Re \left\{ \frac{zh'(z)}{Q(z)} \right\} > 0, z \in U$.

If k is analytic in U with $k(0) = q(0)$, $k(U) \subseteq D$ and

$$\theta(k(z)) + zk'(z)\phi(k(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)), \quad (7)$$

then $k(z) \prec q(z)$ and q is the best dominant of (7).

Lemma 2.3 ([27]). *Let $\xi, \phi \in \mathbb{C}$ with $\phi \neq 0$ and let q be a convex function in U with*

$$\Re \left\{ 1 + \frac{zq''(z)}{q'(z)} \right\} > \max\{0; -\Re \frac{\xi}{\phi}\}.$$

If k is analytic in U and

$$\xi k(z) + \phi zk'(z) \prec \xi q(z) + \phi zq'(z), \quad (8)$$

then $k \prec q$ and q is the best dominant of (8).

Lemma 2.4 ([11]). *Let q be a univalent function in U and let θ and ϕ be analytic in a domain D containing $q(U)$. Suppose that*

(i) $\Re \left\{ \frac{\theta'(q(z))}{\phi(q(z))} \right\} > 0$ for $z \in U$,

(ii) $Q(z) = zq'(z)\phi(q(z))$ is starlike univalent in U .

If $k \in H[q(0), 1] \cap \mathcal{Q}$, with $k(U) \subseteq D$, $\theta(k(z)) + zk'(z)\phi(k(z))$ is univalent in U and

$$\theta(q(z)) + zq'(z)\phi(q(z)) \prec \theta(k(z)) + zk'(z)\phi(k(z)), \quad (9)$$

then $q(z) \prec k(z)$ and q is the best subdominant of (9).

Lemma 2.5 ([20]). *Let q be convex univalent in U and let $\beta \in \mathbb{C}$, with $\Re\{\beta\} > 0$. If $k \in H[q(0), 1] \cap \mathcal{Q}$, $k(z) + \beta zk'(z)$ is univalent in U and*

$$q(z) + \beta zq'(z) \prec k(z) + \beta zk'(z), \quad (10)$$

then $q \prec k$ and q is the best subdominant of (10).

Lemma 2.6 ([25]). *The function $q(z) = (1 - z)^{-2ab}$ ($a, b \in \mathbb{C}^*$) is univalent in U if and only if $|2ab - 1| \leq 1$ or $|2ab + 1| \leq 1$.*

3. Subordinant results

Unless otherwise mentioned, we shall assume in the reminder of this paper that $\lambda \geq 0$, $\ell > -p$, $p \in \mathbb{N}$, $\alpha \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $m \in \mathbb{Z}$ and $z \in U$ and the powers are understood as principle values.

Theorem 3.1. *Let $q(z)$ be univalent in U , with $q(0) = 1$ and suppose that*

$$\Re \left\{ 1 + \frac{z q''(z)}{q'(z)} \right\} > \max \left\{ 0; -\frac{p(p+\ell)}{\lambda} \Re \left(\frac{1}{\alpha} \right) \right\}. \quad (11)$$

If $f(z) \in \mathcal{A}(p)$ such that $\frac{z^p}{\mathcal{J}_p^m(\lambda, \ell) f(z)} \neq 0$ and satisfies the subordination

$$\frac{(p+\alpha)}{p} \left(\frac{z^p}{\mathcal{J}_p^m(\lambda, \ell) f(z)} \right) - \frac{\alpha}{p} \left(\frac{z^p \mathcal{J}_p^{m+1}(\lambda, \ell) f(z)}{(\mathcal{J}_p^m(\lambda, \ell) f(z))^2} \right) \prec q(z) + \frac{\lambda \alpha z q'(z)}{p(p+\ell)}, \quad (12)$$

then

$$\frac{z^p}{\mathcal{J}_p^m(\lambda, \ell) f(z)} \prec q(z)$$

and q is the best dominant of (12).

Proof. Define a function $k(z)$ by

$$k(z) = \frac{z^p}{\mathcal{J}_p^m(\lambda, \ell) f(z)} \quad (z \in U), \quad (13)$$

where $k(z)$ is analytic in U with $k(0) = 1$. By differentiating (13) logarithmically with respect to z , we obtain that

$$\frac{z k'(z)}{k(z)} = p - \frac{z (\mathcal{J}_p^m(\lambda, \ell) f(z))'}{\mathcal{J}_p^m(\lambda, \ell) f(z)}. \quad (14)$$

From (14) and (5), a simple computation shows that

$$\frac{(p+\alpha)}{p} \left(\frac{z^p}{\mathcal{J}_p^m(\lambda, \ell) f(z)} \right) - \frac{\alpha}{p} \left(\frac{z^p \mathcal{J}_p^{m+1}(\lambda, \ell) f(z)}{(\mathcal{J}_p^m(\lambda, \ell) f(z))^2} \right) = k(z) + \frac{\lambda \alpha z k'(z)}{p(p+\ell)},$$

hence the subordination (12) is equivalent to

$$k(z) + \frac{\lambda \alpha z k'(z)}{p(p+\ell)} \prec q(z) + \frac{\lambda \alpha z q'(z)}{p(p+\ell)}.$$

Now, applying Lemma 2.3, with $\varphi = \frac{\lambda \alpha}{p(p+\ell)}$ and $\xi = 1$, the proof is completed. \square

Taking $q(z) = \frac{1+Az}{1+Bz}$ ($-1 \leq B < A \leq 1$) in Theorem 3.1, the condition (11) reduces to

$$\Re \left\{ \frac{1-Bz}{1+Bz} \right\} > \max \left\{ 0; -\frac{p(p+\ell)}{\lambda} \Re \left(\frac{1}{\alpha} \right) \right\}. \quad (15)$$

It is easy to check that the function $\psi(\zeta) = \frac{1-\zeta}{1+\zeta}$, $|\zeta| < |B|$, is convex in U and since $\psi(\bar{\zeta}) = \overline{\psi(\zeta)}$ for all $|\zeta| < |B|$, it follows that the image $\psi(U)$ is convex domain symmetric with respect to the real axis, hence

$$\inf \left\{ \Re \left(\frac{1-Bz}{1+Bz} \right) \right\} = \frac{1-|B|}{1+|B|} > 0. \quad (16)$$

Then the inequality (15) is equivalent to $\frac{|B|-1}{|B|+1} \leq \frac{p(p+\ell)}{\lambda} \Re \left(\frac{1}{\alpha} \right)$, hence, we obtain the following corollary.

Corollary 3.2. *Let $f(z) \in \mathcal{A}(p)$, $-1 \leq B < A \leq 1$ and*

$$\max \left\{ 0; -\frac{p(p+\ell)}{\lambda} \Re \left(\frac{1}{\alpha} \right) \right\} \leq \frac{1-|B|}{1+|B|},$$

then

$$\begin{aligned} \frac{(p+\alpha)}{p} \left(\frac{z^p}{\mathcal{J}_p^m(\lambda, \ell) f(z)} \right) - \frac{\alpha}{p} \left(\frac{z^p \mathcal{J}_p^{m+1}(\lambda, \ell) f(z)}{(\mathcal{J}_p^m(\lambda, \ell) f(z))^2} \right) \\ \prec \frac{1+Az}{1+Bz} + \frac{\lambda \alpha}{p(p+\ell)} \frac{(A-B)z}{(1+Bz)^2}, \end{aligned} \quad (17)$$

implies

$$\frac{z^p}{\mathcal{J}_p^m(\lambda, \ell) f(z)} \prec \frac{1+Az}{1+Bz}$$

and $\frac{1+Az}{1+Bz}$ is the best dominant of (17).

Taking $q(z) = \frac{1+z}{1-z}$ in Theorem 3.1 (or putting $A = 1$ and $B = -1$ in Corollary 3.2), the condition (11) reduces to

$$\frac{p(p+\ell)}{\lambda} \Re \left(\frac{1}{\alpha} \right) \geq 0, \quad (18)$$

hence, we obtain the following corollary.

Corollary 3.3. Let $f(z) \in \mathcal{A}(p)$, assume that (18) holds true and

$$\frac{(p+\alpha)}{p} \left(\frac{z^p}{\mathcal{J}_p^m(\lambda, \ell) f(z)} \right) - \frac{\alpha}{p} \left(\frac{z^p \mathcal{J}_p^{m+1}(\lambda, \ell) f(z)}{(\mathcal{J}_p^m(\lambda, \ell) f(z))^2} \right) \prec \frac{1+z}{1-z} + \frac{\lambda \alpha}{p(p+\ell)} \frac{2z}{(1-z)^2}, \quad (19)$$

then

$$\frac{z^p}{\mathcal{J}_p^m(\lambda, \ell) f(z)} \prec \frac{1+z}{1-z}$$

and $\frac{1+z}{1-z}$ is the best dominant of (19).

Theorem 3.4. Let $q(z)$ be univalent in U , with $q(0) = 1$ and $q(z) \neq 0$ for all $z \in U$, $\eta, \zeta \in \mathbb{C}^*$, $\rho, \tau \in \mathbb{C}$, with $\rho + \tau \neq 0$, $f(z) \in \mathcal{A}(p)$ and suppose that f and q satisfy the next conditions:

$$\frac{(\rho + \tau)z^p}{\rho \mathcal{J}_p^{m+1}(\lambda, \ell) f(z) + \tau \mathcal{J}_p^m(\lambda, \ell) f(z)} \neq 0 \quad (z \in U) \quad (20)$$

and

$$\Re \left\{ 1 + \frac{z q''(z)}{q'(z)} - \frac{z q'(z)}{q(z)} \right\} > 0 \quad (z \in U). \quad (21)$$

If

$$1 + \zeta \eta \left\{ p - \frac{\rho z (\mathcal{J}_p^{m+1}(\lambda, \ell) f(z))' + \tau z (\mathcal{J}_p^m(\lambda, \ell) f(z))'}{\rho \mathcal{J}_p^{m+1}(\lambda, \ell) f(z) + \tau \mathcal{J}_p^m(\lambda, \ell) f(z)} \right\} \prec 1 + \eta \frac{z q'(z)}{q(z)}, \quad (22)$$

then

$$\left(\frac{(\rho + \tau)z^p}{\rho \mathcal{J}_p^{m+1}(\lambda, \ell) f(z) + \tau \mathcal{J}_p^m(\lambda, \ell) f(z)} \right)^\zeta \prec q(z)$$

and q is the best dominant of (22).

Proof. Let

$$g(z) = \left(\frac{(\rho + \tau)z^p}{\rho \mathcal{J}_p^{m+1}(\lambda, \ell) f(z) + \tau \mathcal{J}_p^m(\lambda, \ell) f(z)} \right)^\zeta \quad (z \in U), \quad (23)$$

then $g(z)$ is analytic in U , differentiating $g(z)$ logarithmically with respect to z , we obtain

$$\frac{zg'(z)}{g(z)} = \zeta \left\{ p - \frac{\rho z (\mathcal{J}_p^{m+1}(\lambda, \ell) f(z))' + \tau z (\mathcal{J}_p^m(\lambda, \ell) f(z))'}{\rho \mathcal{J}_p^{m+1}(\lambda, \ell) f(z) + \tau \mathcal{J}_p^m(\lambda, \ell) f(z)} \right\}. \quad (24)$$

Now, using Lemma 2.2 with $\theta(w) = 1$ and $\phi(w) = \frac{\eta}{w}$, then θ is analytic in \mathbb{C} and $\phi(w) \neq 0$ is analytic in \mathbb{C}^* . Also if we let

$$Q(z) = zq'(z)\phi(q(z)) = \eta \frac{zq'(z)}{q(z)},$$

and

$$h(z) = \theta(q(z)) + Q(z) = 1 + \eta \frac{zq'(z)}{q(z)},$$

then, $Q(0) = 0$ and $Q'(0) \neq 0$, and the assumption (3.11) yields that Q is a starlike function in U . From (21) we have

$$\Re \left\{ \frac{zh'(z)}{Q(z)} \right\} = \Re \left\{ 1 + \frac{zq''(z)}{q'(z)} - \frac{zq'(z)}{q(z)} \right\} > 0 \quad (z \in U),$$

then, by using Lemma 2.2, we deduce that the assumption (22) implies $g(z) \prec q(z)$ and the function q is the best dominant of (22). \square

Taking $q(z) = \frac{1+Az}{1+Bz}$ ($-1 \leq B < A \leq 1$), $\rho = 0$ and $\tau = \eta = 1$ in Theorem 3.4, the condition (21) reduces to

$$\left\{ 1 - \frac{2Bz}{1+Bz} - \frac{(A-B)z}{(1+Az)(1+Bz)} \right\} > 0, \quad (25)$$

hence, we obtain the following corollary.

Corollary 3.5. *Let $f(z) \in \mathcal{A}(p)$, assume that (25) holds true, $-1 \leq B < A \leq 1$ and suppose that $\frac{\mathcal{J}_p^m(\lambda, \ell) f(z)}{z^p} \neq 0$ ($z \in U$). If*

$$1 + \zeta \left\{ p - \frac{z (\mathcal{J}_p^m(\lambda, \ell) f(z))'}{\mathcal{J}_p^m(\lambda, \ell) f(z)} \right\} \prec 1 + \frac{(A-B)z}{(1+Az)(1+Bz)}, \quad (26)$$

then

$$\left(\frac{z^p}{\mathcal{J}_p^m(\lambda, \ell) f(z)} \right)^\zeta \prec \frac{1+Az}{1+Bz}, \quad (27)$$

and $\frac{1+Az}{1+Bz}$ is the best dominant of (26).

Putting $p \in \mathbb{N}$, $n = \rho = 0$, $\tau = \eta = 1$ and $q(z) = (1 + Bz)^{\frac{\zeta(A-B)}{B}}$ ($\zeta \in \mathbb{C}^*$, $-1 \leq B < A \leq 1$, $B \neq 0$) in Theorem 3.4 and using Lemma 2.6, it is easy to check that the assumption (21) holds, hence we obtain the next corollary:

Corollary 3.6. *Let $f \in \mathcal{A}(p)$, $\zeta \in \mathbb{C}^*$, $-1 \leq B < A \leq 1$, with $B \neq 0$ and suppose that $\left| \frac{\zeta(A-B)}{B} - 1 \right| \leq 1$ or $\left| \frac{\zeta(A-B)}{B} + 1 \right| \leq 1$. If*

$$1 + \zeta \left(p - \frac{zf'(z)}{f(z)} \right) \prec \frac{1 + [B + \zeta(A-B)]z}{1 + Bz}, \quad (28)$$

then

$$\left(\frac{z^p}{f(z)} \right)^\zeta \prec (1 + Bz)^{\frac{\zeta(A-B)}{B}}$$

and $(1 + Bz)^{\frac{\zeta(A-B)}{B}}$ is the best dominant of (28).

Putting $m = \rho = 0$, $\tau = 1$, $\eta = \frac{1}{ab}$ ($a, b \in \mathbb{C}^*$), $\zeta = a$, and $q(z) = (1 - z)^{-2ab}$ in Theorem 3.4, hence combining this together with Lemma 2.6, we obtain the following corollary.

Corollary 3.7. *Let $f(z) \in \mathcal{A}(p)$, assume that (21) holds true and $a, b \in \mathbb{C}^*$ such that $|2ab - 1| \leq 1$ or $|2ab + 1| \leq 1$. If*

$$1 + \frac{1}{b} \left(p - \frac{zf'(z)}{f(z)} \right) \prec \frac{1 + z}{1 - z}, \quad (29)$$

then

$$\left(\frac{z^p}{f(z)} \right)^a \prec (1 - z)^{-2ab}$$

and $(1 - z)^{-2ab}$ is the best dominant of (29).

Theorem 3.8. *Let $q(z)$ be univalent in U , with $q(0) = 1$, $\eta, \zeta \in \mathbb{C}^*$, $\rho, \tau, \sigma, \varkappa \in \mathbb{C}$, with $\rho + \tau \neq 0$ and $f(z) \in \mathcal{A}(p)$. Suppose that f and q satisfy the next two conditions:*

$$\frac{(\rho + \tau)z^p}{\rho \mathcal{J}_p^{m+1}(\lambda, \ell) f(z) + \tau \mathcal{J}_p^m(\lambda, \ell) f(z)} \neq 0 \quad (z \in U) \quad (30)$$

and

$$\Re \left\{ 1 + \frac{zq''(z)}{q'(z)} \right\} > \max\{0; -\Re\left(\frac{\sigma}{\eta}\right)\} \quad (z \in U). \quad (31)$$

If

$$\mathcal{F}(z) = \left(\frac{(\rho + \tau)z^p}{\rho \mathcal{J}_p^{m+1}(\lambda, \ell) f(z) + \tau \mathcal{J}_p^m(\lambda, \ell) f(z)} \right)^\zeta.$$

$$\cdot \left[\sigma + \zeta \eta \left(p - \frac{\rho z (\mathcal{J}_p^{m+1}(\lambda, \ell) f(z))' + \tau z (\mathcal{J}_p^m(\lambda, \ell) f(z))'}{\rho \mathcal{J}_p^{m+1}(\lambda, \ell) f(z) + \tau \mathcal{J}_p^m(\lambda, \ell) f(z)} \right) \right] + \varkappa \quad (32)$$

and

$$\mathcal{F}(z) \prec \sigma q(z) + \eta z q'(z) + \varkappa, \quad (33)$$

then

$$\left(\frac{(\rho + \tau) z^p}{\rho \mathcal{J}_p^{m+1}(\lambda, \ell) f(z) + \tau \mathcal{J}_p^m(\lambda, \ell) f(z)} \right)^\zeta \prec q(z) \quad (34)$$

and q is the best dominant of (34).

Proof. Let $g(z)$ defined by (23), we see that (24) holds and

$$zg'(z) = \zeta g(z) \left\{ p - \frac{\rho z (\mathcal{J}_p^{m+1}(\lambda, \ell) f(z))' + \tau z (\mathcal{J}_p^m(\lambda, \ell) f(z))'}{\rho \mathcal{J}_p^{m+1}(\lambda, \ell) f(z) + \tau \mathcal{J}_p^m(\lambda, \ell) f(z)} \right\}. \quad (35)$$

Now, Let us consider $\theta(w) = \sigma w + \varkappa$ and $\phi(w) = \eta$, then θ and $\phi(w) \neq 0$ are analytic in \mathbb{C} . Also if we let

$$Q(z) = zq'(z)\phi(q(z)) = \eta zq'(z),$$

and

$$h(z) = \theta(q(z)) + Q(z) = \sigma q(z) + \eta zq'(z) + \varkappa$$

then the assumption (31) yields that Q is a starlike function in U and that

$$\Re \left\{ \frac{zh'(z)}{Q(z)} \right\} = \Re \left\{ \frac{\sigma}{\eta} + 1 + \frac{zq''(z)}{q'(z)} \right\} > 0 \quad (z \in U).$$

The proof follows by applying Lemma 2.2. \square

Taking $q(z) = \frac{1+Az}{1+Bz}$ ($-1 \leq B < A \leq 1$) and using (16), the condition (31) reduces to

$$\max \left\{ 0; -\Re \frac{\sigma}{\eta} \right\} \leq \frac{1-|B|}{1+|B|}, \quad (36)$$

hence, putting $\eta = \rho = 1$ and $\tau = 0$ in Theorem 3.8, we obtain the following corollary.

Corollary 3.9. Let $f(z) \in \mathcal{A}(p)$, $-1 \leq B < A \leq 1$ and $\sigma \in \mathbb{C}$ with

$$\max \{0; -\Re(\sigma)\} \leq \frac{1-|B|}{1+|B|},$$

suppose that $\frac{z^p}{\mathcal{J}_p^{m+1}(\lambda, \ell)f(z)} \neq 0$ ($z \in U$) and let $\zeta \in \mathbb{C}^*$. If

$$\left(\frac{z^p}{\mathcal{J}_p^{m+1}(\lambda, \ell)f(z)}\right)^\zeta \cdot \left[\sigma + \zeta \left(p - \frac{z(\mathcal{J}_p^{m+1}(\lambda, \ell)f(z))'}{\mathcal{J}_p^{m+1}(\lambda, \ell)f(z)}\right)\right] + \varkappa \\ \prec \sigma \frac{1+Az}{1+Bz} + \frac{(A-B)z}{(1+Bz)^2} + \varkappa, \quad (37)$$

then

$$\left(\frac{z^p}{\mathcal{J}_p^{m+1}(\lambda, \ell)f(z)}\right)^\zeta \prec \frac{1+Az}{1+Bz}$$

and $\frac{1+Az}{1+Bz}$ is the best dominant of (37).

Putting $m = \rho = 0, \eta = \tau = 1, p \in \mathbb{N}$ and $q(z) = \frac{1+z}{1-z}$ in Theorem 3.8, we obtain the following corollary.

Corollary 3.10. Let $f(z) \in \mathcal{A}(p)$ such that $\frac{z^p}{f(z)} \neq 0$ for all $z \in U$ and let $\zeta \in \mathbb{C}^*$. If

$$\left(\frac{z^p}{f(z)}\right)^\zeta \cdot \left[\sigma + \zeta \left(p - \frac{zf'(z)}{f(z)}\right)\right] + \varkappa \prec \sigma \frac{1+z}{1-z} + \frac{2z}{(1-z)^2} + \varkappa, \quad (38)$$

then

$$\left(\frac{z^p}{f(z)}\right)^\zeta \prec \frac{1+z}{1-z}$$

and $\frac{1+z}{1-z}$ is the best dominant of (38).

4. Superordination and sandwich results

Theorem 4.1. Let $q(z)$ be convex in U , with $q(0) = 1$ and

$$\frac{\lambda}{p(p+\ell)} \Re(\alpha) > 0. \quad (39)$$

Let $f(z) \in \mathcal{A}(p)$ and suppose that $\frac{z^p}{\mathcal{J}_p^m(\lambda, \ell)f(z)} \in H[q(0), 1] \cap \mathcal{Q}$. If the function

$$\frac{(p+\alpha)}{p} \left(\frac{z^p}{\mathcal{J}_p^m(\lambda, \ell)f(z)}\right) - \frac{\alpha}{p} \left(\frac{z^p \mathcal{J}_p^{m+1}(\lambda, \ell)f(z)}{(\mathcal{J}_p^m(\lambda, \ell)f(z))^2}\right),$$

is univalent in U and

$$q(z) + \frac{\lambda \alpha z q'(z)}{p(p+\ell)} \prec \frac{(p+\alpha)}{p} \left(\frac{z^p}{\mathcal{J}_p^m(\lambda, \ell) f(z)} \right) - \frac{\alpha}{p} \left(\frac{z^p \mathcal{J}_p^{m+1}(\lambda, \ell) f(z)}{(\mathcal{J}_p^m(\lambda, \ell) f(z))^2} \right), \quad (40)$$

then

$$q(z) \prec \frac{z^p}{\mathcal{J}_p^m(\lambda, \ell) f(z)}$$

and q is the best subordinator of (40).

Proof. Let $k(z)$ defined by (13), we see that (14) holds. After some computations, we obtain

$$\frac{(p+\alpha)}{p} \left(\frac{z^p}{\mathcal{J}_p^m(\lambda, \ell) f(z)} \right) - \frac{\alpha}{p} \left(\frac{z^p \mathcal{J}_p^{m+1}(\lambda, \ell) f(z)}{(\mathcal{J}_p^m(\lambda, \ell) f(z))^2} \right) = k(z) + \frac{\lambda \alpha z k'(z)}{p(p+\ell)} \quad (41)$$

and now, by using Lemma 2.5 we obtain the desired result. \square

Taking $q(z) = \frac{1+Az}{1+Bz}$ ($-1 \leq B < A \leq 1$) in Theorem 4.1, we obtain the following corollary.

Corollary 4.2. Let $q(z)$ be convex in U , with $q(0) = 1$ and $\left[\frac{\lambda}{p(p+\ell)} \Re(\alpha) \right] > 0$.

Let $f(z) \in \mathcal{A}(p)$ and suppose that $\frac{\mathcal{J}_p^m(\lambda, \ell) f(z)}{z^p} \in H[q(0), 1] \cap \mathcal{Q}$. If the function

$$\frac{(p+\alpha)}{p} \left(\frac{z^p}{\mathcal{J}_p^m(\lambda, \ell) f(z)} \right) - \frac{\alpha}{p} \left(\frac{z^p \mathcal{J}_p^{m+1}(\lambda, \ell) f(z)}{(\mathcal{J}_p^m(\lambda, \ell) f(z))^2} \right),$$

is univalent in U and

$$\begin{aligned} & \frac{1+Az}{1+Bz} + \frac{\lambda \alpha}{p(p+\ell)} \frac{(A-B)z}{(1+Bz)^2} \\ & \prec \frac{(p+\alpha)}{p} \left(\frac{z^p}{\mathcal{J}_p^m(\lambda, \ell) f(z)} \right) - \frac{\alpha}{p} \left(\frac{z^p \mathcal{J}_p^{m+1}(\lambda, \ell) f(z)}{(\mathcal{J}_p^m(\lambda, \ell) f(z))^2} \right), \quad (42) \end{aligned}$$

then

$$\frac{1+Az}{1+Bz} \prec \frac{z^p}{\mathcal{J}_p^m(\lambda, \ell) f(z)}$$

and $\frac{1+Az}{1+Bz}$ ($-1 \leq B < A \leq 1$) is the best subordinator of (42).

The proof of the following theorem is similar to the proof of Theorem 4.1, so we state the theorem without proof.

Theorem 4.3. Let $q(z)$ be convex in U , with $q(0) = 1$, $\eta, \zeta \in \mathbb{C}^*$, $\rho, \tau \in \mathbb{C}$, with $\rho + \tau \neq 0$. Let $f(z) \in \mathcal{A}(p)$ and satisfy the next conditions:

$$\frac{(\rho + \tau)z^p}{\rho \mathcal{J}_p^{m+1}(\lambda, \ell) f(z) + \tau \mathcal{J}_p^m(\lambda, \ell) f(z)} \neq 0 \quad (z \in U)$$

and

$$\left(\frac{(\rho + \tau)z^p}{\rho \mathcal{J}_p^{m+1}(\lambda, \ell) f(z) + \tau \mathcal{J}_p^m(\lambda, \ell) f(z)} \right)^\zeta \in H[q(0), 1] \cap \mathcal{Q}.$$

If the function $1 + \zeta \eta \left\{ p - \frac{\rho z (\mathcal{J}_p^{m+1}(\lambda, \ell) f(z))' + \tau z (\mathcal{J}_p^m(\lambda, \ell) f(z))'}{\rho \mathcal{J}_p^{m+1}(\lambda, \ell) f(z) + \tau \mathcal{J}_p^m(\lambda, \ell) f(z)} \right\}$ is univalent in U

and

$$1 + \eta \frac{z q'(z)}{q(z)} \prec 1 + \zeta \eta \left\{ p - \frac{\rho z (\mathcal{J}_p^{m+1}(\lambda, \ell) f(z))' + \tau z (\mathcal{J}_p^m(\lambda, \ell) f(z))'}{\rho \mathcal{J}_p^{m+1}(\lambda, \ell) f(z) + \tau \mathcal{J}_p^m(\lambda, \ell) f(z)} \right\},$$

then

$$q(z) \prec \left(\frac{(\rho + \tau)z^p}{\rho \mathcal{J}_p^{m+1}(\lambda, \ell) f(z) + \tau \mathcal{J}_p^m(\lambda, \ell) f(z)} \right)^\zeta \quad (43)$$

and q is the best subordinant of (43).

By applying Lemma 2.4, we obtain the following theorem.

Theorem 4.4. Let $q(z)$ be convex in U , with $q(0) = 1$, $\eta, \zeta \in \mathbb{C}^*$, $\rho, \tau, \sigma, \varkappa \in \mathbb{C}$, with $\rho + \tau \neq 0$ and $\Re \left(\frac{\sigma}{\eta} q'(z) \right) > 0$. Let $f(z) \in \mathcal{A}(p)$ and satisfy the next conditions:

$$\frac{(\rho + \tau)z^p}{\rho \mathcal{J}_p^{m+1}(\lambda, \ell) f(z) + \tau \mathcal{J}_p^m(\lambda, \ell) f(z)} \neq 0 \quad (z \in U)$$

and

$$\left(\frac{(\rho + \tau)z^p}{\rho \mathcal{J}_p^{m+1}(\lambda, \ell) f(z) + \tau \mathcal{J}_p^m(\lambda, \ell) f(z)} \right)^\zeta \in H[q(0), 1] \cap \mathcal{Q}.$$

If the function \mathcal{F} given by (32) is univalent in U and

$$\sigma q(z) + \eta z q'(z) + \varkappa \prec \mathcal{F}(z), \quad (44)$$

then

$$q(z) \prec \left(\frac{(\rho + \tau)z^p}{\rho \mathcal{J}_p^{m+1}(\lambda, \ell) f(z) + \tau \mathcal{J}_p^m(\lambda, \ell) f(z)} \right)^\zeta$$

and q is the best subordinant of (44).

Combining Theorem 3.1 and Theorem 4.1, we obtain the following sandwich theorem.

Theorem 4.5. *Let q_1 and q_2 be two convex functions in U , such that $q_1(0) = q_2(0) = 1$ and $\left[\frac{\lambda}{p(p+\ell)} \Re(\alpha)\right] > 0$. Let $f(z) \in \mathcal{A}(p)$ and suppose that $\frac{\mathcal{J}_p^m(\lambda, \ell)f(z)}{z^p} \in H[q(0), 1] \cap \mathcal{Q}$. If the function*

$$\frac{(p+\alpha)}{p} \left(\frac{z^p}{\mathcal{J}_p^m(\lambda, \ell)f(z)} \right) - \frac{\alpha}{p} \left(\frac{z^p \mathcal{J}_p^{m+1}(\lambda, \ell)f(z)}{(\mathcal{J}_p^m(\lambda, \ell)f(z))^2} \right)$$

is univalent in U and

$$q_1(z) + \frac{\lambda \alpha z q_1'(z)}{p(p+\ell)} \prec \frac{(p+\alpha)}{p} \left(\frac{z^p}{\mathcal{J}_p^m(\lambda, \ell)f(z)} \right) - \frac{\alpha}{p} \left(\frac{z^p \mathcal{J}_p^{m+1}(\lambda, \ell)f(z)}{(\mathcal{J}_p^m(\lambda, \ell)f(z))^2} \right) \prec q_2(z) + \frac{\lambda \alpha z q_2'(z)}{p(p+\ell)}, \quad (45)$$

then

$$q_1(z) \prec \frac{z^p}{\mathcal{J}_p^m(\lambda, \ell)f(z)} \prec q_2(z)$$

and q_1 and q_2 are, respectively, the best subdominant and dominant of (45).

Combining Theorem 3.4 and Theorem 4.3, we obtain the following sandwich theorem.

Theorem 4.6. *Let $q(z)$ be convex in U , with $q(0) = 1$, $\eta, \zeta \in \mathbb{C}^*$, $\rho, \tau \in \mathbb{C}$, with $\rho + \tau \neq 0$. Let $f(z) \in \mathcal{A}(p)$ and satisfy $\frac{(\rho+\tau)z^p}{\rho \mathcal{J}_p^{m+1}(\lambda, \ell)f(z) + \tau \mathcal{J}_p^m(\lambda, \ell)f(z)} \neq 0$ ($z \in U$) and $\left(\frac{(\rho+\tau)z^p}{\rho \mathcal{J}_p^{m+1}(\lambda, \ell)f(z) + \tau \mathcal{J}_p^m(\lambda, \ell)f(z)} \right)^\zeta \in H[q(0), 1] \cap \mathcal{Q}$. If the function*

$$1 + \zeta \eta \left\{ p - \frac{\rho z (\mathcal{J}_p^{m+1}(\lambda, \ell)f(z))' + \tau z (\mathcal{J}_p^m(\lambda, \ell)f(z))'}{\rho \mathcal{J}_p^{m+1}(\lambda, \ell)f(z) + \tau \mathcal{J}_p^m(\lambda, \ell)f(z)} \right\}$$

is univalent in U and

$$1 + \eta \frac{z q_1'(z)}{q_1(z)} \prec 1 + \zeta \eta \left\{ p - \frac{\rho z (\mathcal{J}_p^{m+1}(\lambda, \ell)f(z))' + \tau z (\mathcal{J}_p^m(\lambda, \ell)f(z))'}{\rho \mathcal{J}_p^{m+1}(\lambda, \ell)f(z) + \tau \mathcal{J}_p^m(\lambda, \ell)f(z)} \right\} \prec 1 + \eta \frac{z q_2'(z)}{q_2(z)}, \quad (46)$$

then

$$q_1(z) \prec \left(\frac{(\rho + \tau)z^p}{\rho \mathcal{J}_p^{m+1}(\lambda, \ell) f(z) + \tau \mathcal{J}_p^m(\lambda, \ell) f(z)} \right)^\zeta \prec q_2(z)$$

and q_1 and q_2 are, respectively, the best subordinate and dominant of (46).

Combining Theorem 3.8 and Theorem 4.4, we obtain the following sandwich theorem.

Theorem 4.7. Let q_1 and q_2 be two convex functions in U , with $q_1(0) = q_2(0) = 1$, let $\eta, \zeta \in \mathbb{C}^*$, $\rho, \tau, \sigma, \varkappa \in \mathbb{C}$, with $\rho + \tau \neq 0$ and $\Re\left(\frac{\sigma}{\eta} q'(z)\right) > 0$. Let $f(z) \in \mathcal{A}(p)$ satisfies

$\frac{(\rho + \tau)z^p}{\rho \mathcal{J}_p^{m+1}(\lambda, \ell) f(z) + \tau \mathcal{J}_p^m(\lambda, \ell) f(z)} \neq 0 \quad (z \in U)$ and $\left(\frac{(\rho + \tau)z^p}{\rho \mathcal{J}_p^{m+1}(\lambda, \ell) f(z) + \tau \mathcal{J}_p^m(\lambda, \ell) f(z)} \right)^\zeta \in H[q(0), 1] \cap \mathcal{Q}$. If the function \mathcal{F} given by (32) is univalent in U and

$$\sigma q_1(z) + \eta z q_1'(z) + \chi \prec \mathcal{F}(z) \prec \sigma q_2(z) + \eta z q_2'(z) + \varkappa, \quad (47)$$

then

$$q_1(z) \prec \left(\frac{(\rho + \tau)z^p}{\rho \mathcal{J}_p^{m+1}(\lambda, \ell) f(z) + \tau \mathcal{J}_p^m(\lambda, \ell) f(z)} \right)^\zeta \prec q_2(z)$$

and q_1 and q_2 are, respectively, the best subordinate and dominant of (47).

Remark 4.8. By Specializing λ, ℓ and m in the above results, we obtain the corresponding results for the operators $I_p^m(\lambda, \ell)$, $J_p^m(\lambda, \ell)$, $D_{\lambda, p}^m$ and $D_{\lambda, p}^m$, which are defined in introduction.

Acknowledgments

The authors would like to record their sincere thanks to the referee(s) for their valuable comments and insightful suggestions.

REFERENCES

- [1] R. M. Ali - V. Ravichandran - K. G. Subramanian, *Differential sandwich theorems for certain analytic functions*, Far East J. Math. Sci. 15 (1) (2004), 87–94.
- [2] F. M. Al-Oboudi, *On univalent functions defined by a generalized Salagean operator*, Internat. J. Math. Math. Sci. 27 (2004), 1429–1436.

- [3] M. K. Aouf - F. M. Al-Oboudi - M. M. Haidan, *On some results for λ -spirallike and λ -Robertson functions of complex order*, Publ. Inst. Math. Belgrade 77 (91) (2005), 93–98.
- [4] M. K. Aouf - T. Bulboacă, *Subordination and superordination properties of multivalent functions defined by certain integral operator*, J. Franklin Instit. 347 (2010), 641–653.
- [5] M. K. Aouf - R. M. El-Ashwah - S. M. El-Deeb, *Some inequalities for certain p -valent functions involving an extended multiplier transformations*, Proc. Pakistan Acad. Sci. 46 (4) (2009), 217–221.
- [6] M. K. Aouf - A. O. Mostafa, *On a subclass of $n - p$ -valent prestarlike functions*, Comput. Math. Appl. 55 (2008), 851–861.
- [7] M. K. Aouf - A. O. Mostafa - R. M. El-Ashwah, *Sandwich theorems for p -valent functions defined by a certain integral operator*, Math. Comput. Modelling 53 (2011), 1647–1653.
- [8] M. K. Aouf - T. M. Seoudy, *On differential sandwich theorems of analytic functions defined by generalized Salagean integral operator*, Appl. Math. Letters 24 (2011), 1364–1368.
- [9] T. Bulboacă, *A class of superordination-preserving integral operators*, Indag. Math. (N. S.) 13 (3) (2002), 301–311.
- [10] T. Bulboacă, *Classes of first order differential subordinations*, Demonstratio Math. 35 (2) (2002), 287–292.
- [11] T. Bulboacă, *Differential Subordinations and Superordinations, Recent Results*, House of Scientific Book Publ., Cluj-Napoca, 2005.
- [12] A. Catas, *On certain classes of p -valent functions defined by multiplier transformations*, in Proc. Book of the International Symposium on Geometric Functions Theory and Applications, Istanbul, Turkey, (August 2007), 241–250.
- [13] N. E. Cho - T. H. Kim, *Multiplier transformations and strongly close-to-convex functions*, Bull. Korean Math. Soc. 40 (3) (2003), 399–410.
- [14] N. E. Cho - H. M. Srivastava, *Argument estimates of certain analytic functions defined by a class of multiplier transformations*, Math. Comput. Modelling 37 (1-2) (2003), 39–49.
- [15] R. M. El-Ashwah - M. K. Aouf, *Some properties of new integral operator*, Acta Univ. Apulensis 24 (2010), 51–61.
- [16] T. M. Flett, *The dual of an identity of Hardy and Littlewood and some related inequalities*, J. Math. Anal. Appl. 38 (1972), 746–765.
- [17] M. Kamali - H. Orhan, *On a subclass of certian starlike functions with negative coefficients*, Bull. Korean Math. Soc. 41 (1) (2004), 53–71.
- [18] S. S. Kumar - H. C. Taneja - V. Ravichandran, *Classes multivalent functions defined by Dziok-Srivastava linear operaor and multiplier transformations*, Kyungpook Math. J. 46 (2006), 97–109.

- [19] S. S. Miller - P. T. Mocanu, *Differential Subordination: Theory and Applications*, Series on Monographs and Textbooks in Pure and Applied Mathematics 225, Marcel Dekker Inc., New York and Basel, 2000.
- [20] S. S. Miller - P. T. Mocanu, *Subordinates of differential subordinations*, Complex Variables 48 (10) (2003), 815–826.
- [21] H. Orhan - H. Kiziltunc, *A generalization on subfamily of p -valent functions with negative coefficients*, Appl. Math. Comput. 155 (2004), 521–530.
- [22] J. Patel, *Inclusion relations and convolution properties of certain subclasses of analytic functions defined by generalized Salagean operator*, Bull. Belg. Math. Soc. Simon Stevin 15 (2008), 33–47.
- [23] J. Patel - P. Sahoo, *Certain subclasses of multivalent analytic functions*, Indian J. Pure Appl. Math. 34 (2003), 487–500.
- [24] J. K. Prajapat, *Subordination and superordination preserving properties for generalized multiplier transformation operator*, Math. Comput. Modelling 55 (2012), 1456–1465.
- [25] W. C. Royster, *On the univalence of a certain integral*, Michigan Math. J. 12 (1965), 385–387.
- [26] G. S. Salagean, *Subclasses of univalent functions*, Lecture Notes in Math. 1013, Springer-Verlag (1983), 362–372.
- [27] T. N. Shanmugam - V. Ravichandran - S. Sivasubramanian, *Differential sandwich theorems for some subclasses of analytic functions*, J. Austr. Math. Anal. Appl. 3 (1) (2006), Art. 8, 1–11.
- [28] H. M. Srivastava - M. K. Aouf - R. M. El-Ashwah, *Some inclusion relationships associated with a certain class of integral operators*, Asian European J. Math. 3 (4) (2010), 667–684.
- [29] H. M. Srivastava - K. Suchithra - B. Adolf Stephen - S. Sivasubramanian, *Inclusion and neighborhood properties of certain subclasses of multivalent functions of complex order*, J. Ineq. Pure Appl. Math. 7 (5) (2006), Art. 191, 1–8.

RABHA M. EL-ASHWAH

Department of Mathematics, Faculty of Science

Damietta University

New Damietta 34517, Egypt

e-mail: r_elashwah@yahoo.com

MOHAMED K. AOUF

Department of Mathematics, Faculty of Science

Mansoura University

Mansoura 35516, Egypt

e-mail: mkaouf127@yahoo.com

ALI SHAMANDY

Department of Mathematics, Faculty of Science

Mansoura University

Mansoura 35516, Egypt

e-mail: shamandy16@hotmail.com

SHEZA M. EL-DEEB

Department of Mathematics, Faculty of Science

Damietta University

New Damietta 34517, Egypt

e-mail: shezaeldeeb@yahoo.com